Some Aspects of the Measurement of the Q Factor of Transmission Lines*

P. I. SOMLO†

Summary—This paper deals with two basic problems in the measurement of the Q factor of low-loss transmission lines: 1) long lines and 2) short lines with appreciable direct coupling between the driving and pickup probes.

The results are given as relative ordinates on the (distorted) resonance curve which correspond to the correct values of $\pm\frac{1}{2}\delta l$ or $\pm\frac{1}{2}\delta f$ defined by $Q=f/\delta f=l/\delta l$.

I. Introduction

A TRANSMISSION LINE can be described by its characteristic impedance Z_0 (ohms) and its propagation constant $\gamma=\alpha+j\beta$, where α (nepers/m) is the attenuation constant and β (radians/m) is the phase-change constant.

This paper is concerned with the measurement of the Q factor of a transmission line which is defined generally by

$$Q = \frac{2\pi(\text{Energy stored})}{\text{Energy lost per cycle}}$$

and is related to the components of the propagation constant by the simple, exact and approximate expressions given later. As it is often possible to determine the Q factor accurately by a simple resonance technique, the measurement of Q, together with some easily obtainable additional information, provides a convenient alternative method of determining the attenuation constant while avoiding difficulties of matching to the line impedance.

In the case of "short" uniform transmission lines it is standard practice to measure the above defined Q factor by alternative methods.[1–3] One of these is the "line length variation method" in which the line is terminated in short or open circuits, and then $Q=l/\delta l$ where $l(m)$ is the length of the transmission line at resonance and $\frac{1}{2}\delta l(m)$ is the length which is a necessary variation on l to reduce the maximum value of the current (or the voltage) at the termination to its $1/\sqrt{2}$ value.[4]

As far as driving and pickup are concerned there are two practical possibilities. The driving and pickup probes can be at the same, or at opposite ends.

It is obvious from a simple qualitative reasoning that the expression of $Q=l/\delta l$ will be increasingly in error as the transmission line is made longer. When the pickup and feeding probes are at the same end, and the line is made very long, the input impedance approaches the characteristic impedance of the line, and a change in line length will produce hardly any change at the input. Consequently, resonance cannot be detected. From this it is clear that as the length of the line is gradually increased, the resonance curve detected at the input will undergo a distortion until it finally becomes almost a constant.

If the driving and pickup probes are on opposite ends of the line, it is clear that on increasing the length of the line the change at the pickup probe will be approximately exponential. The "familiar" resonance curve will suffer a progressive distortion into an exponential curve.

The first part of this paper deals with the corrections needed to determine the accurate value of the Q factor of the transmission line irrespective of its length.

The second part deals with a special case of the first. It has been found on several occasions while carrying out Q-factor measurements on transmission lines that when both the driving and pickup loops were placed at the same end of the line, there was a noticeable distortion in the shape of the resonance curve, as shown in Fig. 1.

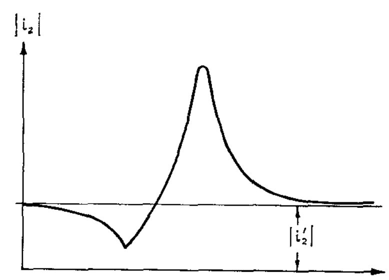

Fig. 1—Shape of resonance curve with fixed coupling between input and output.

* Received January 7, 1963; revised manuscript received June 17, 1963.

† Division of Applied Physics, National Standards Lab., Commonwealth Scientific and Industrial Research Organization, Sydney, Australia.

[1] R. W. King, "General amplitude relations for current and voltage," in "Transmission Line Theory," McGraw-Hill Book Company, Inc., New York, N. Y., ch. 4, pp. 266–286; 1955.

[2] C. G. Montgomery, "The measurement of attenuation," in "Technique of Microwave Measurements," McGraw-Hill Book Company, Inc., New York, N. Y., ch. 13, pp. 821–823; 1947.

[3] M. Wind and H. Rapaport, "Measurement of Q," in "Handbook of Microwave Measurements," Polytechnic Press, Brooklyn, N. Y., Sect. 5.1, pp. 5-1–5-4; 1954.

[4] R. A. Chipman, "A resonance curve method for the absolute measurements of impedance at frequencies of the order 300 Mc/second," *J. Appl. Phys.*, vol. 10, pp. 27–38; January, 1939.

This distortion is caused by the presence of a fixed coupling between the driving and pickup probes.

As it is sometimes very difficult to remove this unwanted coupling, particularly in small diameter coaxial lines, methods of finding the true Q factor from the distorted resonance curve are given below.

II. Theory

A. Measurement of the Q factor of Long Transmission Lines by the Resonance Method

In the following it will be assumed that:

1) the sending end may be regarded as a zero-impedance voltage source,
2) the line is terminated in ideal loss-free short circuits,
3) there is no fixed coupling between input and output,
4) the input and output coupling has been reduced to such a low value that further reduction has no noticeable effect on the resonance curve.

In addition, the approximations are made that resonance occurs when $\beta l = n\pi$ (see Appendix-A) and that $Q=\beta/2\alpha$ (see Appendix-B).

1) *Case a:* Input and output at the same end.

Let us consider a transmission line with ideal short circuits at both ends. (See Fig. 2, using "In" and "Out 1" only.)

For that line

$$\frac{V_1}{i_1} = Z_{\text{in}} = Z_0 \tanh \gamma l$$

where

$$\tanh \gamma l = \frac{\sinh 2\alpha l + j \sin 2\beta l}{\cosh 2\alpha l + \cos 2\beta l}.$$

Since the detector is sensitive to the absolute value of current flowing in the short circuit

$$|i_1| = \frac{|V_1|}{|Z_{\text{in}}|} = \frac{|V_1|}{|Z_0|} \cdot \frac{\cosh 2\alpha l + \cos 2\beta l}{\sqrt{(\sinh^2 2\alpha l + \sin^2 2\beta l)}}. \quad (1)$$

At resonance (when $|i_1|$ is maximum) $l = l_r = n\pi/\beta$ where $n = 1, 2, 3, \cdots$.

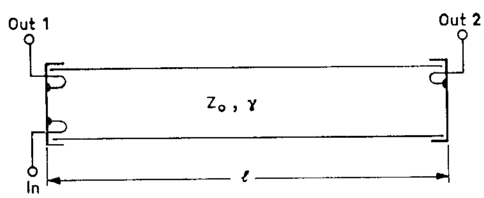

Fig. 2—Resonant transmission line. Case a, coupling at the same end ("Out 1") and Case b, coupling at opposite ends "Out 2".

With this value of l, (1) becomes

$$|i_1|_{\max} = \frac{|V_1|}{|Z_0|} \cdot \frac{\cosh 2x + 1}{\sinh 2x} \quad (2)$$

where $x = n\pi\alpha/\beta = \alpha l_r$ which is the total attenuation of the line, in nepers.

Our aim here is to find the values of currents $|i_1|_{Q\pm}$ which correspond to the changed lengths of the resonating line $l_{Q\pm} = l_r \pm (1/2)\delta l$ which by definition yield the value of Q in the relation $Q = l_r/\delta l$.

It can be shown that this current is

$$|i_1|_{Q\pm} = \frac{|V_1|}{|Z_0|} \frac{\cosh 2y + \cos 2x}{\sqrt{(\sinh^2 2y + \sin^2 2x)}} \quad (3)$$

where $y = x[(1 \pm 1/(2Q)]$, by the use of the relation $Q = \beta/(2\alpha)$.

It is convenient in practice to normalize these currents to the maximum current so that we obtain

$$\frac{|i_1|_{Q\pm}}{|i_1|_{\max}} = \frac{(\cosh 2y + \cos 2x) \sinh 2x}{[\sqrt{(\sinh^2 2y + \sin^2 2x)}](1 + \cosh 2x)}. \quad (4)$$

In Fig. 3, $|i_1|_{Q\pm}/|i_1|_{\max}$ has been plotted against x (nepers) or, more conveniently, against $(20/ln\ 10)$ (db) and $(n/Q) \cdot 10^2$ for different values of Q.

Case b: Input and output at opposite ends.

(The network is shown in Fig. 2, using "In" and "Out 2" only.)

The defining equations of the asymmetrical linear parameters of a transmission line are

$$i_1 = a i_2 + b V_2 \quad (5)$$

$$V_1 = c i_2 + a V_2 \quad (6)$$

where $a = \cosh \gamma l$, $b = (1/Z_0) \sinh \gamma l$, $c = Z_0 \sinh \gamma l$.

Because the termination is a perfect short circuit, $V_2 = 0$.

$$\frac{V_1}{i_1} = Z_{\text{in}} = Z_0 \tanh \gamma l. \quad (7)$$

From (5) and (7)

$$i_2 = V_1/(Z_0 \sinh \gamma l)$$
$$= V_1/[Z_0(\sinh \alpha l \cdot \cos \beta l + j \cosh \alpha l \sin \beta l)]$$

and

$$|i_2| = \frac{|V_1|}{|Z_0|} \frac{1}{\sqrt{(\sinh^2 \alpha l \cos^2 \beta l + \cosh^2 \alpha l \sin^2 \beta l)}}. \quad (8)$$

At resonance when $l = l_r = n/\pi\beta$ after similar steps as before in obtaining (2), $|i_2|_{\max} = |V_1|/|Z_0| \sinh x$. The absolute value of the current which corresponds to $l_{Q\pm}$ is from (8),

$$|i_2|_{Q\pm} = \frac{|V_1|}{|Z_0|} \frac{1}{\sqrt{(\sinh^2 y \cdot \cos^2 x + \cosh^2 y \cdot \sin^2 x)}}. \quad (9)$$

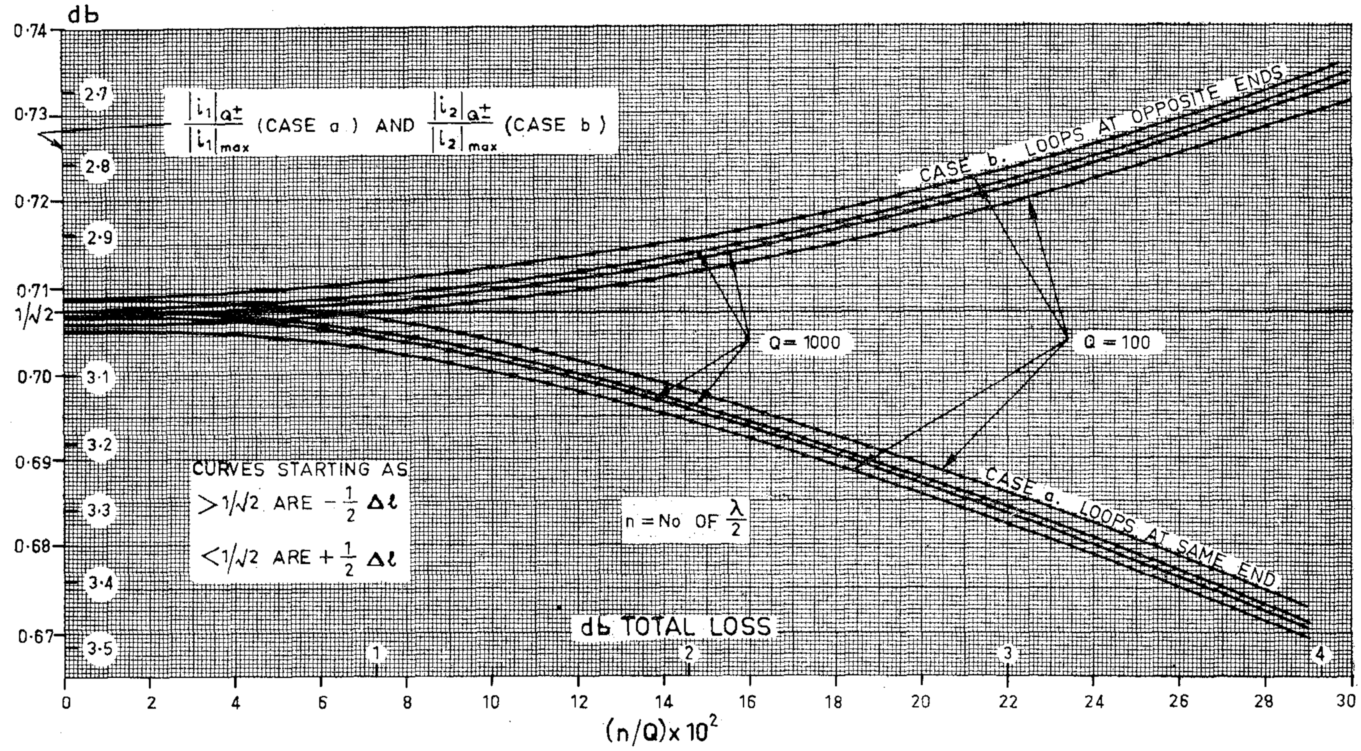

Fig. 3—Total loss in line vs normalized ordinates on the resonance curve which correspond to $\mp\frac{1}{2}\Delta l$. For Case a and again for Case b in each pair of curves at a given Q, the upper correspond to $-\frac{1}{2}\Delta l$ and the lower to $+\frac{1}{2}\Delta l$.

Normalizing as before with respect to the maximum current,

$$\frac{|i_2|_{Q\pm}}{|i_2|_{\max}} = \frac{\sinh x}{\sqrt{(\sinh^2 y \cos^2 x + \cosh^2 y \sin^2 x)}}. \quad \text{(See Fig. 3.)} \quad (10)$$

a) *Evaluation of results:* The plotted results indicate that, when measuring the Q factor of a uniform transmission line by the "length variation method," the familiar concept of detuning until the ordinate on the resonance curve drops by 3.01 db relative to its local maximum is only an approximation. For any given value of total loss in the line the charts give the ordinate corresponding to the value of $\frac{1}{2}\delta l$ needed to calculate Q.

In practice, of course, neither the total loss in the line nor the value of the Q factor is known, in fact, that is to be determined. So the procedure is to detune the line to produce a 3 db-drop, giving an approximate value of the Q factor. This value is used to find a new ordinate (instead of 3 db) which will give a new value of δl, and Q factor. This successive approximation has to be carried out until the change in the Q values is less than the resolution of the measurement. Since the correction is small for practical values of attenuation, the convergence is fast, and two to three steps are adequate in most cases.

Furthermore, Fig. 3 confirms the well-known fact that the measured resonance curve is not symmetrical about its maximum, because, by varying the length of the line, the total attenuation is varied also and an exponential change is superimposed on the resonance curve. Fig. 3 has been plotted for up to 4 db-total loss. For larger amounts of total loss it becomes more practical to carry out a direct attenuation measurenent for several reasons (broadening of resonance curve, matching less critical, etc.).

However, it is interesting to note that the curve representing Case a (probes at the same end) reaches a minimum at about 6 db-total loss and starts to rise again. The reason is that δl becomes so large that $l\mp\frac{1}{2}\delta l$ takes a value on the next, neighbouring resonance curve.

b) *Measurement of the Q factor by frequency variation:* It is shown in Appendix-C that, by assuming that β varies linearly with frequency, the results of calculations based on the "line length variation method" are precisely duplicated.

In the case of non-TEM modes (*e.g.* waveguides), the variation of β vs f is not linear, but having sufficiently high Q values in the relatively small band of δf, the change of β can be approximated by a linear function. It is to be remembered that in waveguides where the total attenuation is small $Q=f/\delta f=(l/\delta l)(\lambda_g/\lambda)^2$ where λ_g is the guide wavelength.[5]

[5] E. L. Ginzton, "Measurement of Attenuation," "Microwave Measurements," McGraw-Hill Book Company, Inc., New York, N. Y., ch. 11.4; 1957.

B. Measurement of the Q Factor in the Presence of Coupling Between the Input and Output Probes

In this section it is assumed (as in Section II-A, assumption 4) that the input and output coupling is adjusted to be loose enough so that there is no significant external loading of the resonating line.

It has been shown in the literature[6] that the input impedance of a short-circuited lossy line plotted on the complex impedance plane traces a spiral if the length of the line is varied. The rate of the contraction of the spiral is a function of the losses present in the line. It can be shown that this rate, or proportional change in diameter after one full revolution ($\lambda/2$), is approximately $\pi^2/(8Q^2)$. Thus if $Q>100$ one revolution on the spiral can be taken as being a circle, and this approximation will be close enough for present purposes.

Considering the line short-circuited at both ends with the two inductive probes positioned close to one of the short circuits with coupling between the probes, and coupling between the probes and the line, the following is the lumped circuit equivalent (see Fig. 4).

It can be shown that the current i_2 consists of two components.

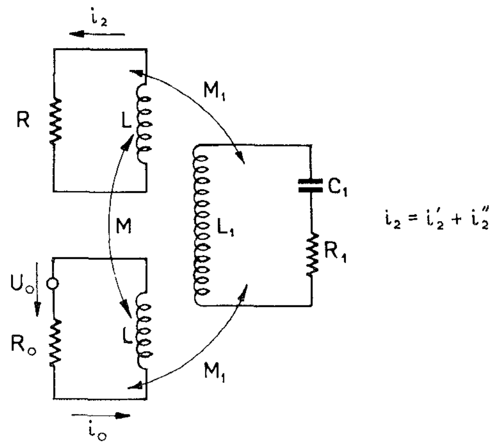

Fig. 4—Lumped circuit equivalent of resonant transmission line with fixed coupling between input and output.

$$i_2 = i_2' + i_2'' = i_0\left\{\frac{j\omega M}{R_T + j\omega L} + \frac{-\omega^2 M_1^2}{R_T R_1 - \omega L\left(\omega L_1 - \dfrac{1}{\omega C_1}\right) + j\left[R_1\omega L + R_T\left(\omega L_1 - \dfrac{1}{\omega C_1}\right)\right]}\right\}. \tag{11}$$

One of these components is not affected by tuning of the circuit, being a constant determined by the mutual coupling (M) between the probes. As will be shown, the other component will change phase with respect to the first component from 0 to π as the $L_1C_1R_1$ circuit is tuned. When the $L_1C_1R_1$ circuit is at resonance these two components will be in phase quadrature. Then $\omega L_1 = 1/(\omega C_1)$ so that

$$i_2' = i_0\frac{\omega^2 ML + j\omega MR_T}{R_T^2 + \omega^2 L}$$

and

$$i_2'' = -i_0\omega^2 M_1^2\frac{R_T R_1 - j\omega R_1 L}{R_T^2 R_1^2 + R_1^2\omega^2 L^2}.$$

The tangents of the phase angles of i_2' and i_2'', respectively, are $\tan\angle i_2' = R_T/(\omega L)$; $\tan\angle i_2'' = -\omega L/R_T$. Since these two quantities are negative reciprocals of each other, the currents i_2' and i_2'' are in phase quadrature when the $L_1C_1R_1$ circuit is at resonance.

By remembering that for the short-circuited lossy transmission line driven by a zero-impedance constant voltage generator, the locus of the current flowing in the short circuit (and in the pickup loop) is a circle on the complex plane as the line length is varied, and remembering the phase relationship between the two components of the current in the pickup loop, i_2 can be represented as in Fig. 5.

Treating the problem as a purely geometrical one, the absolute value of i_2 can be expressed as

$$|i_2| = \sqrt{[i_2'^2 \pm 2i_2'\sqrt{(-s^2+2s)} + 2s]}$$

if the system is normalized so that the radius of the circle is unity.

To find the apparent peak of the "resonance" curve

$$\frac{\partial |i_2|}{\partial s} = 0 \quad\text{and}\quad s = 1 \pm \sqrt{\frac{1}{i_2'^2+1}}.$$

Substituting this value of s (with the positive sign) into $|i_2|$ we get

$$|i_2|_{\max} = \sqrt{\left\{(i_2'^2+1)\left[1 + 2\sqrt{\left(\frac{1}{i_2'^2+1}\right)}\right] + 1\right\}}. \tag{12}$$

Remembering that the aim is to measure the Q factor of the resonating transmission line the question is: what values will $|i_2|$ take when the line is detuned (by the shift of the position of the short circuit at the far end) so that $|i_2''|$ drops to $|i_2''|_{\max}/\sqrt{2}$. For brevity let these values of $|i_2|$ be called $|i_2|_Q$. From the geometry

$$|i_2|_Q = \sqrt{(i_2'^2 \pm 2i_2' + 2)}. \tag{13}$$

[6] J. C. Slater, "Transmission lines," in "Microwave Transmission," McGraw-Hill Book Company, Inc., New York, N. Y., ch. 1, pp. 33–37; 1942.

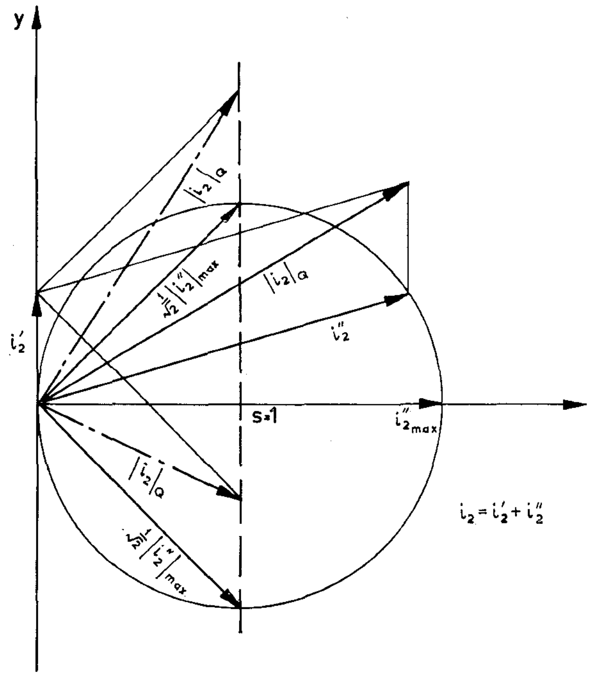

Fig. 5—Vector diagram of currents in a resonant transmission line with fixed coupling between input and output.

During the measurement the only reference level is the maximum of $|i_2|$ (the peak of the distorted resonance curve), so it is necessary to relate $|i_2|_Q$ to that maximum.

$$\frac{|i_2|_Q}{|i_2|_{\max}} = \frac{i_2'^2 \pm 2i_2' + 2}{\sqrt{\left\{(i_2'^2+1)\left[1+2\sqrt{\left(\frac{1}{i_2'^2+1}\right)}\right]+1\right\}}}. \tag{14}$$

On the other hand, to find the magnitude of i_2' the line has to be tuned far off resonance (see Fig. 1). Again, this value must be related to the apparent maximum. Therefore,

$$\frac{|i_2'|}{|i_2|_{\max}} = \frac{|i_2'|}{\sqrt{\left\{(i_2'^2+1)\left[1+2\sqrt{\left(\frac{1}{i_2'^2+1}\right)}\right]+2\right\}}}. \tag{15}$$

In Fig. 6 $|i_2|_Q/|i_2|_{\max}$ has been plotted against $|i_2'|/|i_2|_{\max}$. This chart is used in the following way.

Detune the line sufficiently to find the value of the asymptote and relate this value to the apparent maximum. Take the corresponding values of $|i_2|_Q/|i_2|_{\max}$ from the chart, noting that the smaller ratios apply to the steeper side of the resonance curve. Detune the line until the readings drop to the two given fractions of the maximum. These are the positions of the sliding short circuit that will give the correct value of the Q factor in the formula $Q=l/\delta l$ where δl is the distance between the two positions of the short circuit and l is the full

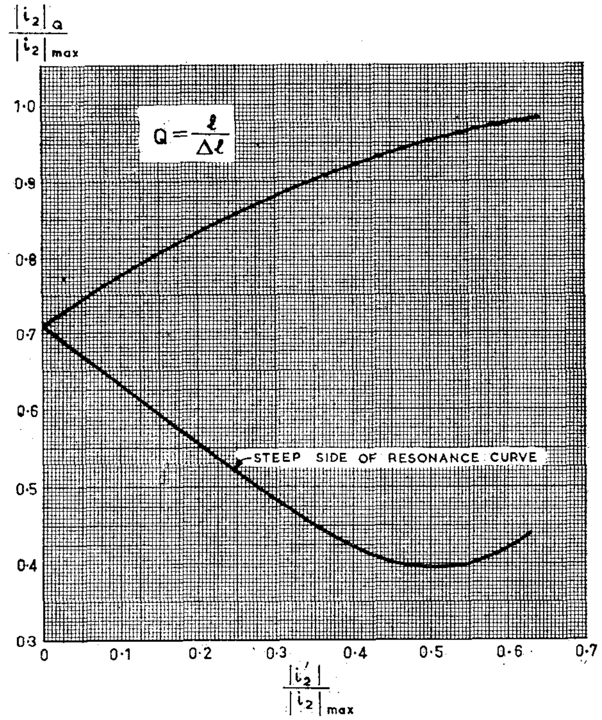

Fig. 6—Normalized value of the asymptote of the distorted resonance curve (see Fig. 1) vs normalized ordinates on the resonance curve which correspond to $\pm\frac{1}{2}\Delta l$ in the case of fixed coupling between the input and output loops.

length of the line. The position of the short circuit which corresponds to the line at resonance will be the mean of the two positions used in the Q-factor measurement.

III. Conclusions

Two aspects of Q-factor measurements of transmission lines have been discussed which lead to corrections to be applied to the familiar method of the "3 db-drop." It has been shown that corrections are necessary in the measurement of the Q factor of a "long" transmission line having appreciable loss between its terminating short circuits. It has been shown also that accurate Q measurements may be made in the presence of fixed coupling between the driving and pickup probes.

The corrections to be used are given as relative amplitudes normalized to the maximum of the "resonance" curve, which correspond to the true values of $\pm\frac{1}{2}\delta l$ or $\pm\frac{1}{2}\delta f$ in the Q-factor formula $Q=f/\delta f$ or $Q=l/\delta l$.

Appendix

A. Definition of Resonance

In the case of a transmission line short-circuited at both ends with both the driving and pickup probes at the same end, resonated by variation of the line length, the maximum current in the pickup probe will occur when the absolute value of the input impedance reaches a minimum. For that line

$$|Z_{\text{in}}| = |Z_0| \frac{\sqrt{(\sinh^2 2\alpha l + \sin^2 2\beta l)}}{\cosh 2\alpha l + \cos 2\beta l}$$

let

$$\beta l = n(\pi + \delta)$$

where n is the integral number of half wavelengths in the line and δ is a small quantity to be evaluated. In that case $2\beta l = 2n(\pi + \delta)$ and $2\alpha l = 2\alpha n(\pi + \delta)/\beta = n(\pi+\delta)/Q$.

$$|Z_{in}| = |Z_0| \frac{\sqrt{\left\{\left[\sinh\frac{n\pi}{Q}\cosh\frac{n\delta}{Q} + \cosh\frac{n\pi}{Q}\sinh\frac{n\delta}{Q}\right]^2 + [\sin 2n\pi\cos 2n\delta + \cos 2n\pi \sin 2n\delta]^2\right\}}}{\cosh\frac{n\pi}{Q}\cosh\frac{n\delta}{Q} + \sinh\frac{n\pi}{Q}\sinh\frac{n\delta}{Q} + \cos 2n\pi\cos 2n\delta - \sin 2n\pi \sin 2n\delta}$$

(see Appendix-B)

assuming that $\delta \ll 1$ and $n/Q \ll 1$ and taking only the first-order terms into account

$$|Z_{in}| \approx |Z_0| \frac{n}{2}\sqrt{\left(4\delta^2 + \frac{2\pi}{Q^2}\delta + \frac{\pi^2}{Q^2}\right)},$$

$$|Z_{in}|_{min} \qquad \text{when } \frac{\partial |Z_{in}|}{\partial \delta} = 0 \approx 8\delta + \frac{2\pi}{Q^2}$$

from which $\delta \approx -\pi/(4Q^2)$ and therefore

$$\beta l = n(\pi + \delta) \approx n\pi\left(1 - \frac{1}{4Q^2}\right).$$

As this paper deals with transmission lines having Q factors of the order of hundreds or higher, the statement that at resonance $\beta l = n\pi$ is justified.

It can be shown that the result obtained here is in agreement with that obtained when the two probes are situated at opposite ends of the line.

B. Accuracy of $\beta/(2\alpha) = Q$ for the General Transmission Line

Defining $Q_s = \omega L/R$ and $Q_p = \omega C/G$ it can be shown that the over-all Q factor of the line will be

$$Q = \frac{Q_s Q_p}{Q_s + Q_p}.$$

On the other hand, using the definition of $\gamma = \alpha + j\beta = \sqrt{[(R+j\omega L)(G+j\omega C)]}$ it can be shown, by a calculation too lengthy for inclusion in this paper, that

$$\frac{\beta}{2\alpha} = \frac{Q_s Q_p}{Q_s + Q_p}\left[1 + \frac{(Q_s - Q_p)^2}{4Q_s^2 Q_p^2} - \frac{(Q_s^2 - Q_p^2)^2}{(4Q_s^2 Q_p^2)^2} + \cdots\right].$$

The series in the brackets indicates that $\beta/(2\alpha)$ does not give the precise value of the over-all Q factor except in the special case when $Q_s = Q_p$ (distortion-less line). The largest divergence occurs if there is only one kind of loss when the series takes the form

$$1 + \frac{1}{4Q^2} - \frac{1}{16Q^4} + \cdots.$$

Taking such a case as the extreme and specifying the Q_s or $Q_p > 100$ (while Q_p or $Q_s = \infty$, respectively)

$$\frac{1}{Q}\frac{\beta}{2\alpha} - 1 < 0.000025.$$

This shows that the approximation $\beta/(2\alpha) = Q$ is sufficiently accurate.

C. Measurement of the Q Factor by Frequency Variation

It will be shown that by assuming that β varies linearly with frequency ($\beta = 2\pi f/v$) for $|i|_Q/|i|_{max}$ precisely the same relationship applies as established for the "line length variation method" irrespective of the form of variation of α with frequency.

The relationships to be used are

$$\beta = 2\pi f/v; \quad l_0 = v/(2f_0) = \lambda/2; \quad l = nl_0; \quad \alpha = k(f)$$

$$\alpha = \beta/(2\alpha) = \pi f/[vk(f)] = f_0/\delta f = 1/\Delta f$$

$$f_Q = f_0 \mp \tfrac{1}{2}\delta f = f_0(1 \mp \tfrac{1}{2}\Delta f) = f_0(1 \mp 1/(2Q)$$

$$x = n\pi/(2Q); \qquad y = x[1 \mp 1/(2Q)].$$

When the input and output probes are at the same end, it was shown that

$$|i_1| = \frac{|V_1|}{|Z_0|}\frac{\cos 2\beta l_0 n + \cosh 2\alpha l_0 n}{\sqrt{(\sin^2 2\beta l_0 n + \cosh^2 2\alpha l_0 n)}}$$

$$= \frac{|V_1| \quad \cos\frac{2l_0 2\pi f}{v}n + \cosh 2l_0 nk(f)}{|Z_0|\sqrt{\left[\sin^2\frac{2l_0 2\pi f}{v}n + \sinh^2 2l_0 nk(f)\right]}}.$$

The value of $|i_1|$ if $f = f_Q$ is

$$|i_1| = \frac{|V_1|}{|Z_0|}\frac{\cos\frac{4\pi n l_0}{v}f_Q + \cosh 2nl_0 k(f_Q)}{\sqrt{\left[\sin^2\frac{4\pi n l_0}{v}f_Q + \sinh^2 2nl_0 k(f_Q)\right]}}$$

since

$$\cos\left[\frac{4\pi n l_0}{v}f_0\left(1 \mp \frac{1}{2Q}\right)\right] = \cos\frac{2\pi l_0}{v}\frac{n}{Q}\frac{v}{2l_0} = \cos\frac{n\pi}{Q}$$

$$= \cos 2x$$

and

$$\sin\left[\frac{4\pi n l_0}{v} f_0\left(1 \mp \frac{1}{2Q}\right)\right] = \mp \sin\frac{n\pi}{Q} = \mp \sin 2x$$

and

$$2nl_0k(f_Q) = \frac{2\pi l_0}{v}\frac{n}{Q}f_0\left(1 \mp \frac{1}{2Q}\right) = \frac{n\pi}{Q}\left(1 \mp \frac{1}{2Q}\right) = 2y$$

$$|i_1| = \frac{|V_1|}{|Z_0|}\frac{\cos 2x + \cosh 2y}{\sqrt{(\sin^2 2x + \sinh^2 2y)}}$$

which is the same function as was obtained from the "line length variation method." Similarly, it can be shown that when the two probes are at opposite ends the result is identical, irrespective of whether the line length or the frequency has been changed to obtain the value of the Q factor.

The reason for this is that $|i|_Q/|i|_{max}=f(x)$ where $x=n\pi/(2Q)$ (nepers) which means that the correction to be used in the Q-factor measurement is a function of the total attenuation only, and therefore the same result will be obtained by either method of measurement.

A Nonuniform Coaxial Line with an Isoperimetric Sheath Deformation*

N. SESHAGIRI†

Summary—**For impedance matching in transmission lines, nonuniform lines, obeying laws of taper like the exponential, the Dolph-Chebyshev etc., are used. For the nonuniform coaxial line, constructional advantages can be derived for the same electrical performance if it has a uniform circular inner conductor with an outer conductor having an isoperimetric transition, from circular to elliptic cross section, in conformity with the established laws of taper. This problem has been examined in the paper, and the required design formulas as well as the design charts are developed. The effect of an impedance and geometric discontinuity at the low-impedance junction of such a nonuniform line and the concentric circular uniform line is discussed. The use of the isoperimetric transition line in microwave components is indicated.**

I. Introduction

A COMMON PROBLEM of systems design is impedance matching. In coaxial transmission lines, this presents difficulties associated with electrical and mechanical design considerations. A solution to the mechanical aspect of the problem has been attempted by the use of nonuniform transmission lines, with a variation of the diameter of either the inner or the outer conductor satisfying the electrical requirements. The latter approach is suggestive of an alternative method wherein the outer conductor transforms isoperimetrically from a circular cross section to an elliptic cross section, the inner conductor being uniformly circular. Mechanical advantages can be derived for the same electrical performance if the increase of ellipticity is related to the existing laws of transition. The proposed structure permits easy and continuous installation of any law of taper. Thus, for instance, the difficulty encountered to install the exponential law[1] or the Orlov's law[2] of taper is not appreciably more than that for a linear taper. The design can also be used for tapered terminations at microwaves,[3] and in the design of microwave components where work on the use of nonuniform lines has been reported.[4,5]

To develop the required design formulas it is first necessary to carry out the field analysis of infinitely long uniform lines with a circular cylindrical inner conductor surrounded by an elliptic outer conductor. Morse and Feshbach[6] give an expression for the case of an inner conductor in the form of a thin wire. A similar analysis can also be made using a Schwarz's transformation.[7] The requirement of the design considered in this paper being that of an inner conductor whose radius is comparable to the dimensions of the ellipse, a different

* Received March 11, 1963; revised manuscript received July 23, 1963.

† Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore, India.

[1] C. R. Burrows, "The exponential transmission line," *Bell Sys. Tech. J.*, vol. 17, pp. 555–573; October, 1938.

[2] S. I. Orlov, "Concerning the theory of non-uniform transmission lines," *J. Tech. Phys. USSR*, vol. 26, pp. 2361–2372; October, 1956. (Translated by APS, vol. 1, pp. 2284–2294; October, 1957.)

[3] G. T. Clemens, "A tapered line termination at microwaves," *Quart. J. Appl. Math.*, vol. 7, pp. 425–432; January, 1950.

[4] C. P. Womack, "The use of exponential transmission lines in microwave components," IRE Trans. on Microwave Theory and Techniques, vol. MTT-10, pp. 124–132; March, 1962.

[5] R. N. Ghose, "Exponential transmission lines as resonators and transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 213–217; July, 1957.

[6] P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," McGraw-Hill Book Co., Inc., New York, N. Y., p. 1203; 1953.

[7] H. A. Schwarz, "Notizia sulla rappresentazione conforme di un'area ellittica sopra un'area circolare," *Annali di Matematica (II)*, vol. 3, pp. 166–173; 1869.